https://www.linkedin.com/feed/update/urn:li:activity:6609718826240221185

(**a**) It is known that $x,y,z \geq 0$ and $x+y+z = \frac{1}{2}$. Prove that

$$\frac{1-x}{1+x} \cdot \frac{1-y}{1+y} \cdot \frac{1-z}{1+z} \geq \frac{1}{3}.$$

(**b**) It is known that $x_1,x_2,\ldots,x_n \geq 0$ and $x_1+x_2+\ldots+x_n = \frac{1}{2}$. Prove that

$$\frac{1-x_1}{1+x_1} \cdot \frac{1-x_2}{1+x_2} \cdot\ldots\cdot \frac{1-x_n}{1+x_n} \geq \frac{1}{3}.$$

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Since (**a**) is particular case of (**b**) we will solve (**b**) by preliminarily reformulated it in more convenient equivalent form. Namely, by replacing $(x_1,x_2,\ldots,x_n)$ with $\left(\frac{x_1}{2},\frac{x_2}{2},\ldots,\frac{x_n}{2}\right)$ we obtain instead $\sum_{k=1}^{n} x_k = \frac{1}{2}$ and $\prod_{k=1}^{n} \frac{1-x_k}{1+x_k} \geq \frac{1}{3}$, respectively, $\sum_{k=1}^{n} x_k = 1$ and $\prod_{k=1}^{n} \frac{2-x_k}{2+x_k} \geq \frac{1}{3}$.

Lemma. For any $x,y \geq 0$ such that $x+y \leq 1$ holds inequality

(**1**) $$\frac{2-x}{2+x} \cdot \frac{2-y}{2+y} \geq \frac{2-(x+y)}{2+(x+y)}.$$

Proof.

Let $p := x+y$ and $q := xy$. Then $\frac{2-x}{2+x} \cdot \frac{2-y}{2+y} = \frac{4-2p+q}{4+2p+q}$ and since $\frac{4-2p+q}{4+2p+q} = 1-\frac{4p}{4+2p+q}$ increase in $q \geq 0$ then $\frac{4-2p+q}{4+2p+q} \geq \frac{4-2p}{4+2p} = \frac{2-p}{2+p}$.

Thus, $\frac{2-x}{2+x} \cdot \frac{2-y}{2+y} \geq \frac{2-(x+y)}{2+(x+y)}$.

We will prove that $x_1,x_2,\ldots,x_n \geq 0$ such that $\sum_{k=1}^{n} x_k \leq 1$ holds inequality

(**2**) $$\prod_{k=1}^{n} \frac{2-x_k}{2+x_k} \geq \frac{2-\sum_{k=1}^{n} x_k}{2+\sum_{k=1}^{n} x_k}.$$

Having inequality (**1**) (by replacing (x,y) with (x_1,x_2)) as base of Math Induction for any $x_1,x_2,\ldots,x_n,x_{n+1} \geq 0$ such that $\sum_{k=1}^{n+1} x_k \leq 1$ and, denoting $p := \sum_{k=1}^{n} x_k \leq 1$, we obtain $\prod_{k=1}^{n} \frac{2-x_k}{2+x_k} \geq \frac{2-p}{2+p}$. Since $\sum_{k=1}^{n+1} x_k = p+x_{n+1}$ then applying inequality (**1**) to $(x,y) = (p,x_{n+1})$ we obtain $\frac{2-p}{2+p} \cdot \frac{2-x_{k+1}}{2+x_{k+1}} \geq \frac{2-(p+x_{n+1})}{2+(p+x_{n+1})} \Leftrightarrow \prod_{k=1}^{n+1} \frac{2-x_k}{2+x_k} \geq \frac{2-\sum_{k=1}^{n+1} x_k}{2+\sum_{k=1}^{n+1} x_k}$.

In particular if $\sum_{k=1}^{n} x_k = 1$ then (**2**) becomes $\prod_{k=1}^{n} \frac{2-x_k}{2+x_k} \geq \frac{2-1}{2+1} = \frac{1}{3}$.